

# Matrix elements for D-meson mixing from 2+1 lattice QCD

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# Outline

Neutral-meson mixing

FNAL/MILC *D*-meson mixing analysis

Correlator analysis

Chiral-cont. extrap + Error analysis

Outlook

# Flavor physics on the lattice

Testing Standard Model through high precision

$$\left( \begin{array}{ccc} |V_{ud}| & |V_{us}| & |V_{ub}| \\ \pi \rightarrow \ell \nu & K \rightarrow \ell \nu & B \rightarrow \tau \nu \\ n \rightarrow p e^- \bar{\nu} & K \rightarrow \pi \ell \nu & B \rightarrow \pi \ell \nu \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ D \rightarrow \ell \nu & D_s \rightarrow \ell \nu & B \rightarrow D \ell \nu \\ D \rightarrow \pi \ell \nu & D \rightarrow K \ell \nu & B \rightarrow D^* \ell \nu \\ |V_{td}| & |V_{ts}| & |V_{tb}| \\ B_0 \text{ mixing} & B_s \text{ mixing} & \text{no hadrons} \end{array} \right)$$

Standard Model parameters (total 26):

Gauge coupling, Yukawa coupling (quark and lepton masses),

**CKM** and PMNS matrix elements, Higgs v.e.v. (EWSB scale),

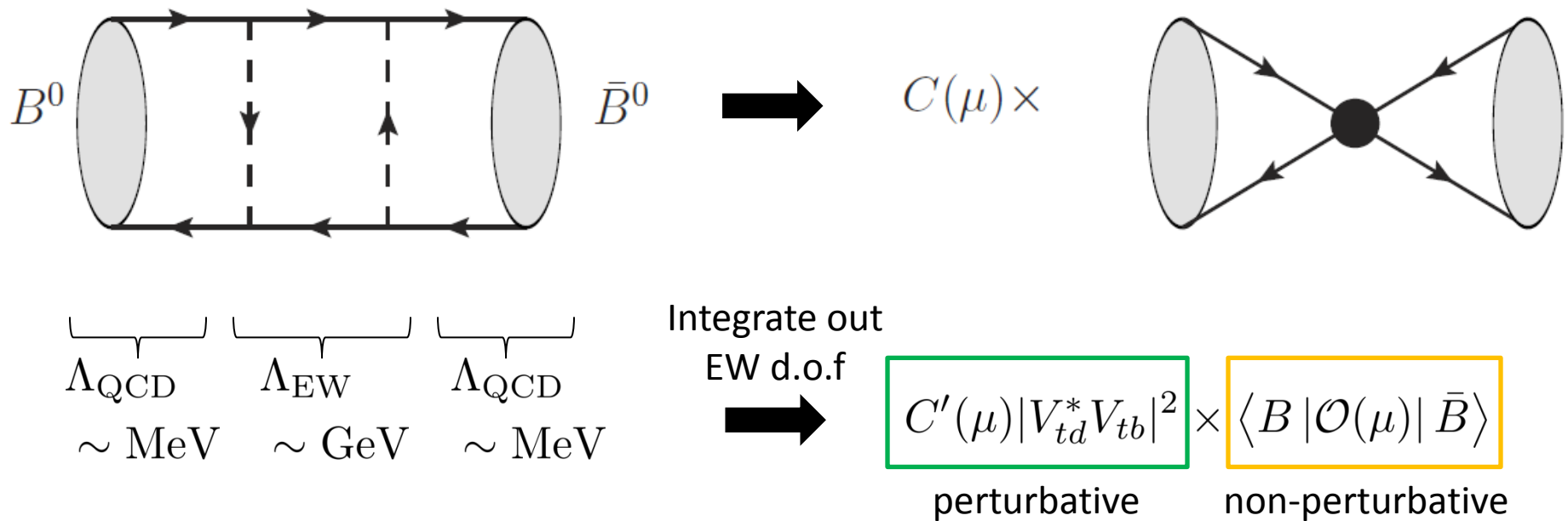
Higgs mass,  $\theta_W$ ,  $\theta_{QCD}$

# Neutral-meson mixing

Separation of scale:

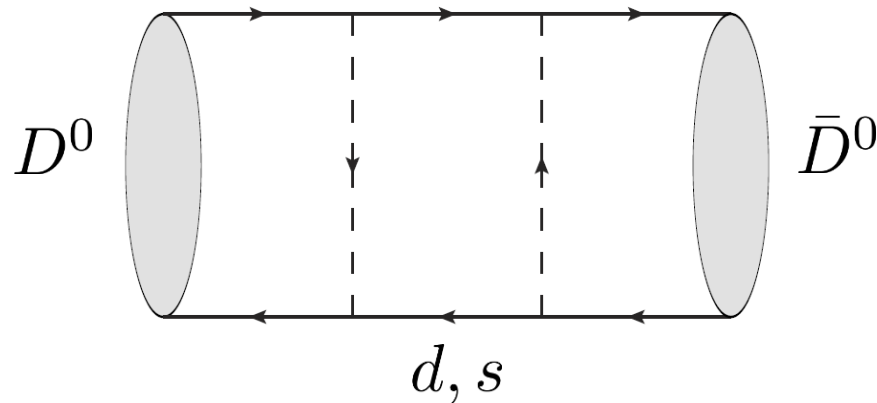
QCD calculation is independent of high energy theory

High energy theory = electroweak (SM) or new physics (BSM)



At hadronic scale, QCD is non-perturbative

# Standard Model short-distance



Up-type quark mixing (unlike Kaon and  $B$ -meson)

**CKM suppressed**

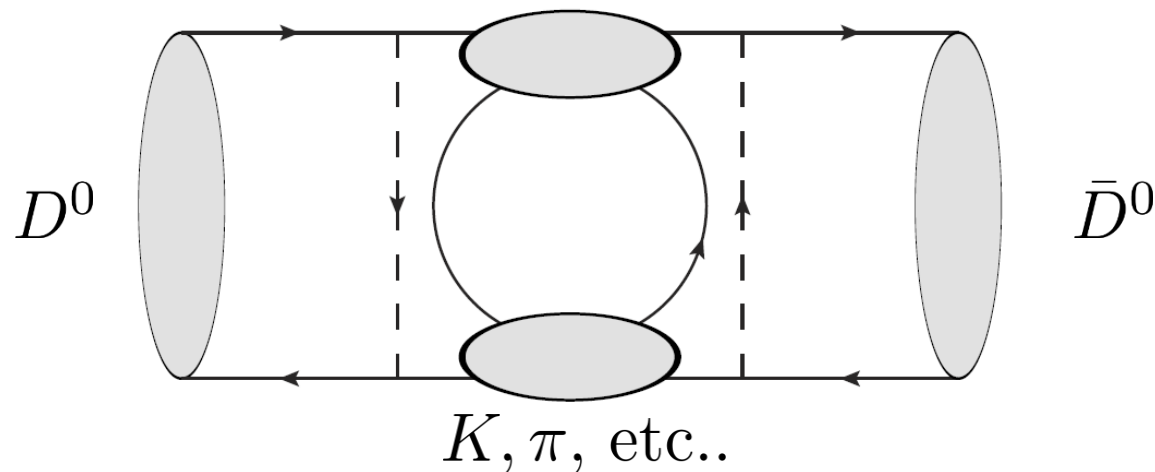
- b-quark suppressed by  $|V_{ub} V_{cb}^*|^2 \sim 0.2^{10}$

**GIM suppressed**

- d- and s-quark diagrams cancel in flavor SU(3) limit

**Very small contribution** (unlike Kaon and  $B$ -meson)

# Standard Model long-distance



Proceeds via on-shell states

Although b-quark CKM sup. @  $0.2^{10}$

- No obvious GIM suppression
- Hadronize via d- and s-quark

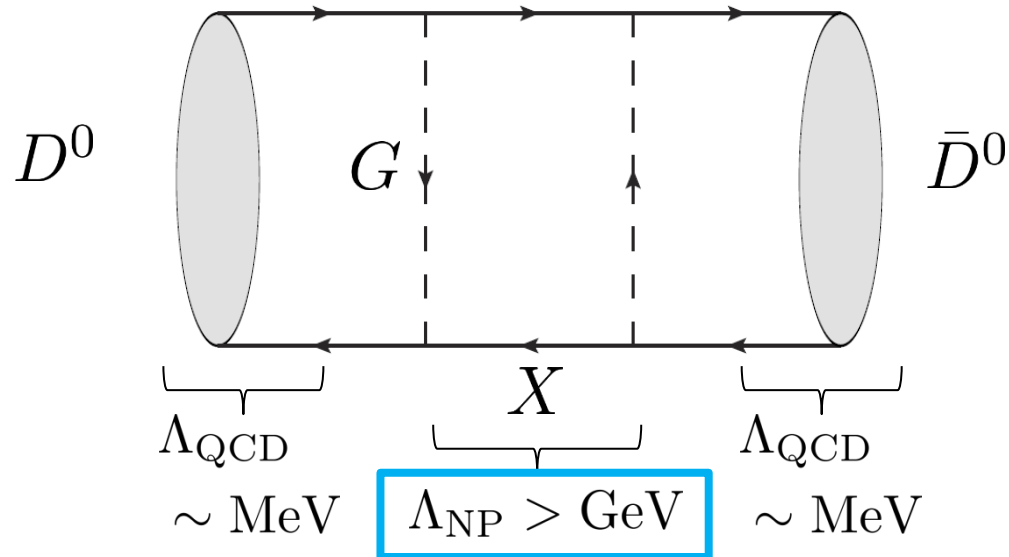
Via pions:  $|V_{cd}V_{ud}^*|^2 \sim 0.2^2$

Via kaons:  $|V_{cs}V_{us}^*|^2 \sim 0.2^2$

“In qualitative accord with experiment”

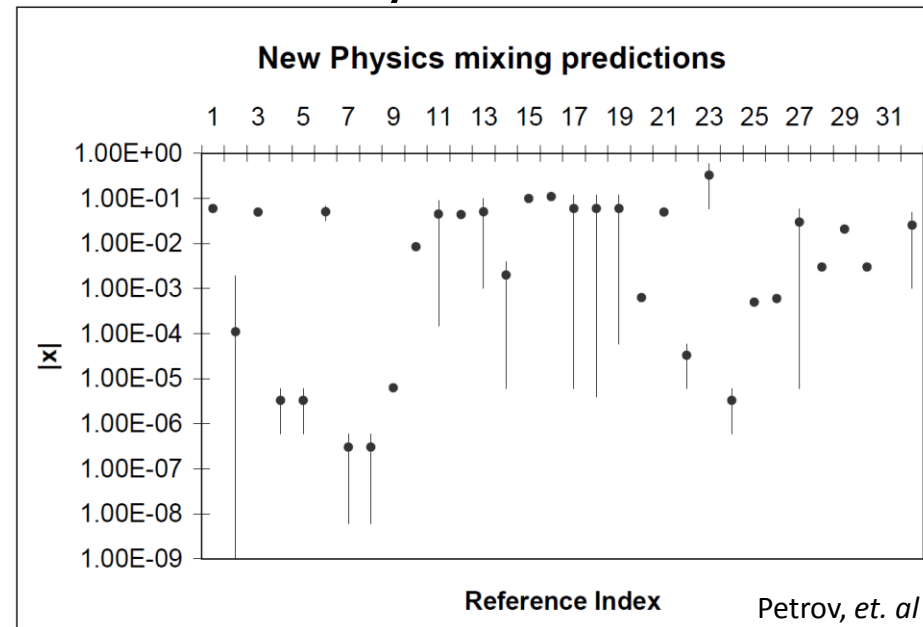
**Possibly dominant**

# BSM contribution



BSM enters in short-distance only

- Possibly BSM dominant
- Many BSM models, some receive strongest constraint from  $D$ -mixing



# Mixing operators

$$\Delta m = C_{\text{NP}}^i \langle D | \mathcal{O}_i | \bar{D} \rangle$$

## Basis of 4-quark operators

$$\mathcal{O}_1 = \bar{\Psi}^a \gamma^\mu L \psi^a \bar{\Psi}^b \gamma^\mu L \psi^b$$

SM (V-A) current

$$\mathcal{O}_2 = \bar{\Psi}^a L \psi^a \bar{\Psi}^b L \psi^b$$

$$\mathcal{O}_3 = \bar{\Psi}^a L \psi^b \bar{\Psi}^b L \psi^a$$

$$\mathcal{O}_4 = \bar{\Psi}^a L \psi^a \bar{\Psi}^b R \psi^a$$

NP only. Right-handed

$$\mathcal{O}_5 = \bar{\Psi}^a L \psi^b \bar{\Psi}^b R \psi^a$$

Only 5 matrix elements. Model independent.

# BSM mixing and experiment

$$\Delta M = 0.0044(20)[\text{ps}^{-1}]$$

$\sim 45\%$

[1402.1664v1]

Exp. err.  $\sim 10\%$  error  
by  $\sim 2020$

$$\begin{aligned}\Delta M &= \langle D | \mathcal{H}_{\text{NP}} | \bar{D} \rangle \\ &= C_{\text{NP}}^i \langle D | \mathcal{O}_i | \bar{D} \rangle\end{aligned}$$

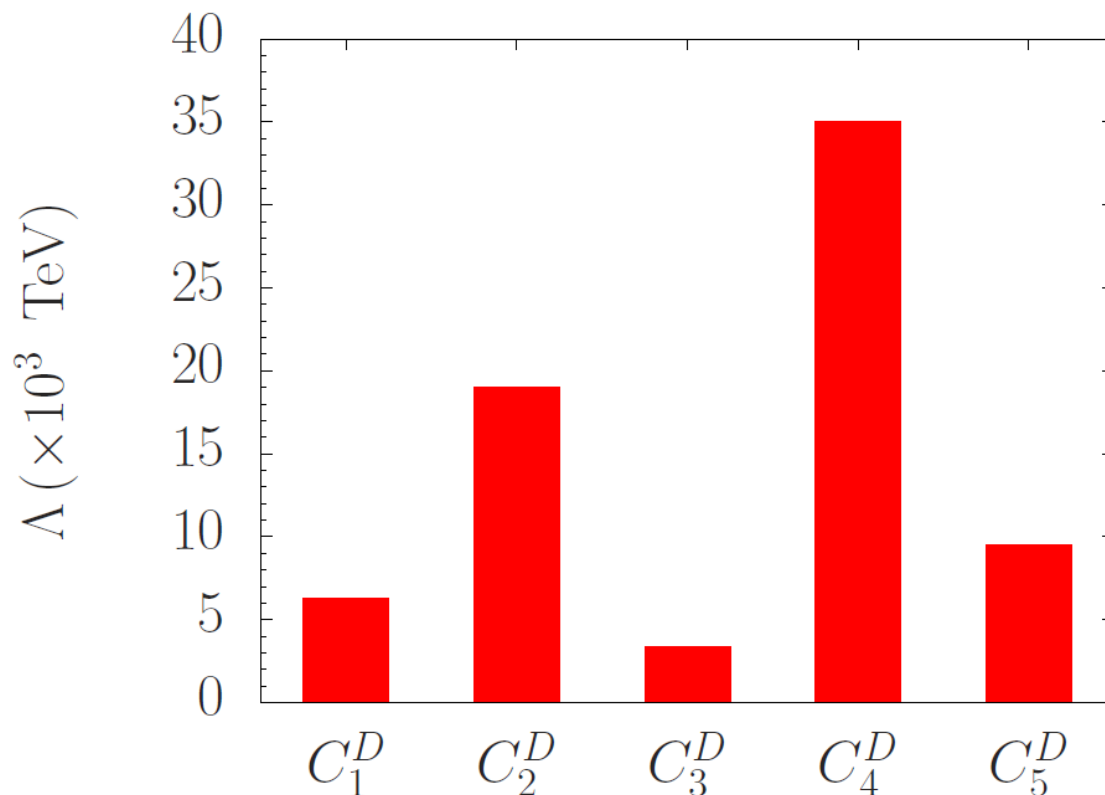
$$C_{\text{NP}}^i = \frac{F_i L_i}{\Lambda^2}$$

$$F_i \sim L_i \sim 1$$

$F_i$  Flavor structure

$L_i$  Loop factor

NP scale lower bound



[1403.7302]

# FNAL/MILC

## *D*-meson mixing analysis

Correlator analysis

Data  
Correlator fits  
Renormalization

# Lattice actions

## Gluon action

$O(a^2)$  improved. Errors start at  $O(\alpha_s a^2, a^4)$ .

## Light-quark action (valence and sea)

$O(a^2)$  improved. Errors start at  $O(\alpha_s a^2, a^4)$ .

Preserve chiral symmetry.

Have spurious taste degrees-of-freedom.

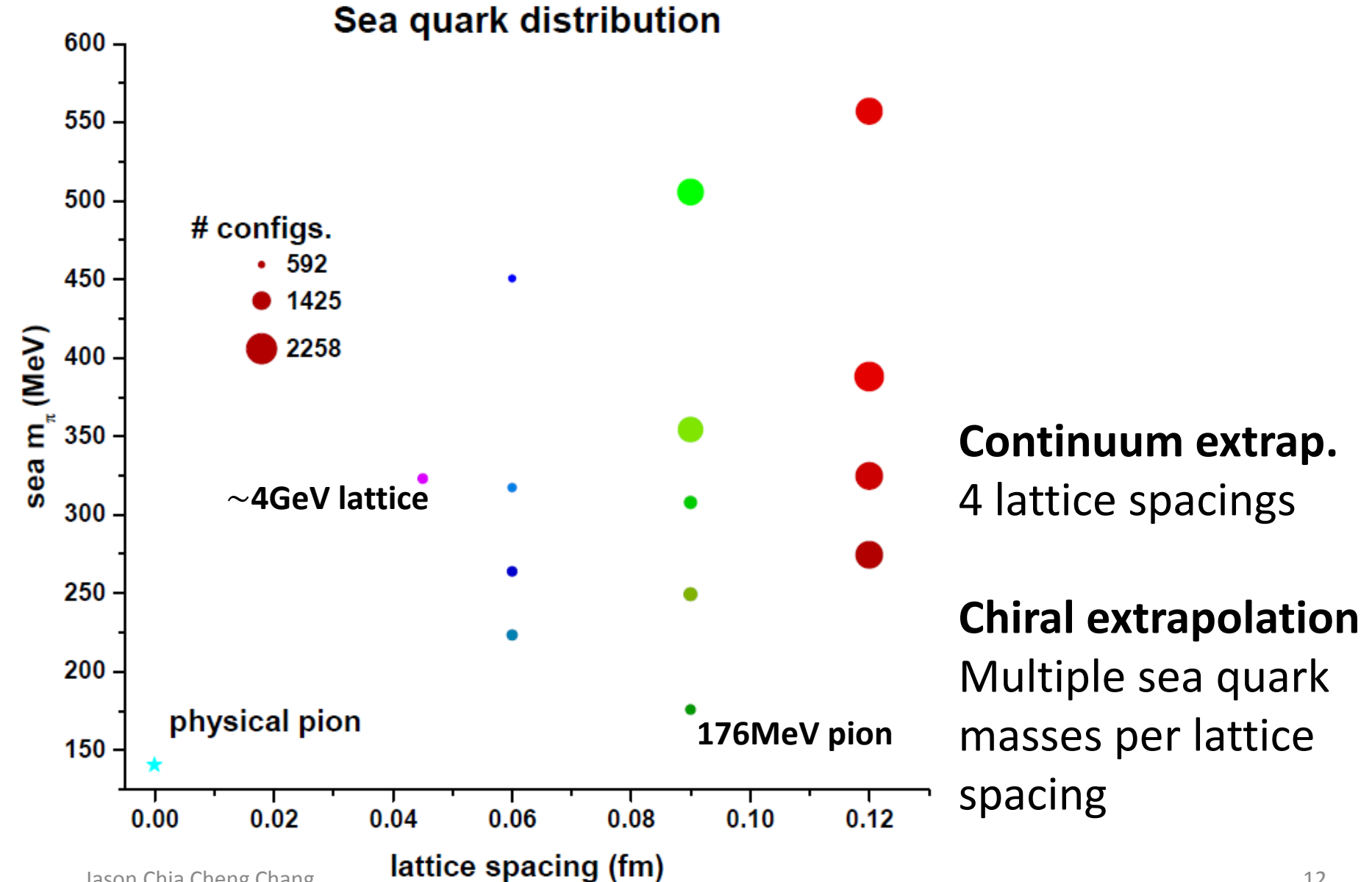
## Heavy-quark action (valence)

$O(a)$  improved. Errors start at  $O(\alpha_s a, a^2)$ .

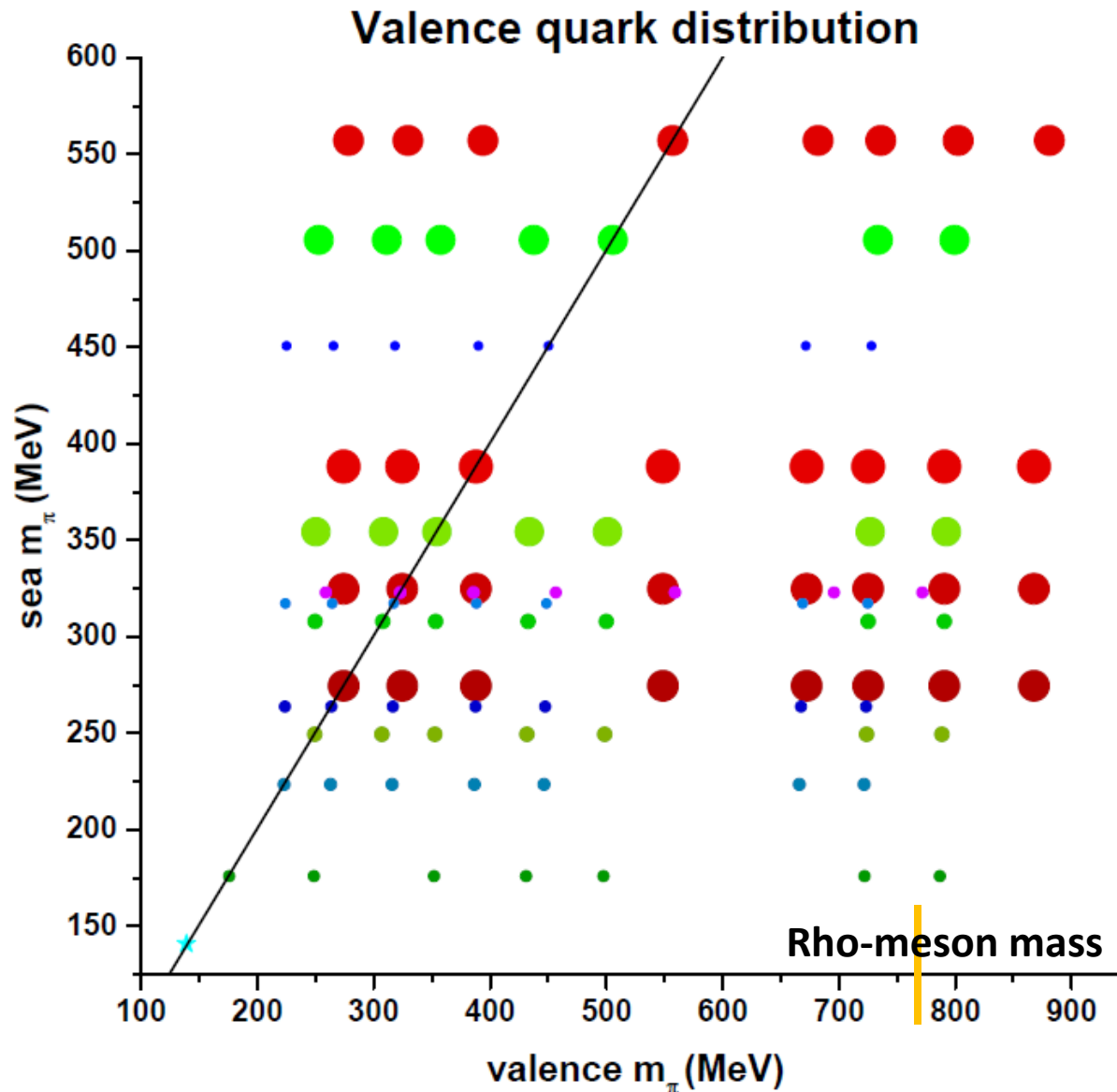
Destroys chiral symmetry.

No spurious taste degrees-of-freedom.

# MILC gauge configurations



# Light-quark propagators



**Partially-quenched  
chiral extrapolation**  
7 to 8 valence masses  
Highly correlated\*

**Other parameters**

Spatial box size

$$m_\pi L \gtrsim 4$$

Temporal length

$$2 \times \text{spatial } L$$

# Heavy-quark propagators

Improved Wilson fermion on Staggered sea

Set charm quark mass to  $\sim m_c$

→ unknown (small) tuning error fixed later

One charm quark mass per gauge configuration

# Correlation functions

## D-meson lattice operators

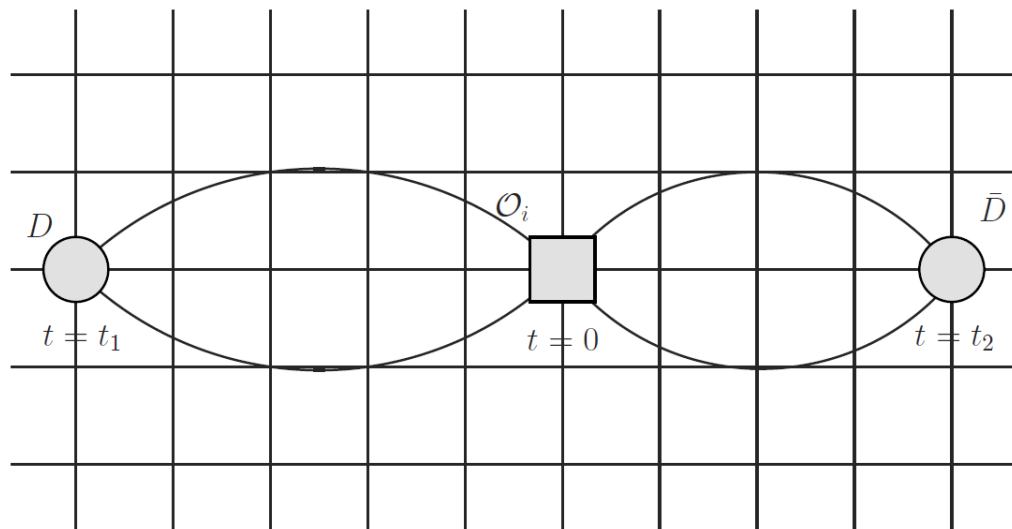
$$D(x) = \bar{\psi} \gamma_5 \Psi(x)$$

$$\bar{D}(x) = \bar{\Psi} \gamma_5 \psi(x)$$

## Correlators

$$C^{2\text{pt}}(t, 0) = \sum_{\mathbf{x}} \langle T \{ \bar{D}(x) D(0) \} \rangle$$

$$C_i^{3\text{pt}}(t_1, t_2, 0) = \sum_{\mathbf{x}_1, \mathbf{x}_2} \langle T \{ D(x_2) \mathcal{O}_i(0) D(x_1) \} \rangle$$



## Fit functions

$$C^{2\text{pt}}(t) = \sum_n (-1)^{n(t+1)} \frac{Z_n^\dagger Z_n}{2E_n} \left( e^{-E_n t} + e^{-E_n (T-t)} \right)$$

$$C_i^{3\text{pt}}(t_1, t_2) = \sum_{m, n} (-1)^{n(t_2+1)} (-1)^{m(|t_1|+1)} \frac{\langle n | \mathcal{O}_i | m \rangle Z_n^\dagger Z_m}{4E_n E_m} e^{-E_n t_2} e^{-E_m |t_1|}$$

# Fit correlation functions (i)

Understand data by staring at it

$$E_0 \approx \ln C(t)/C(t+1)$$

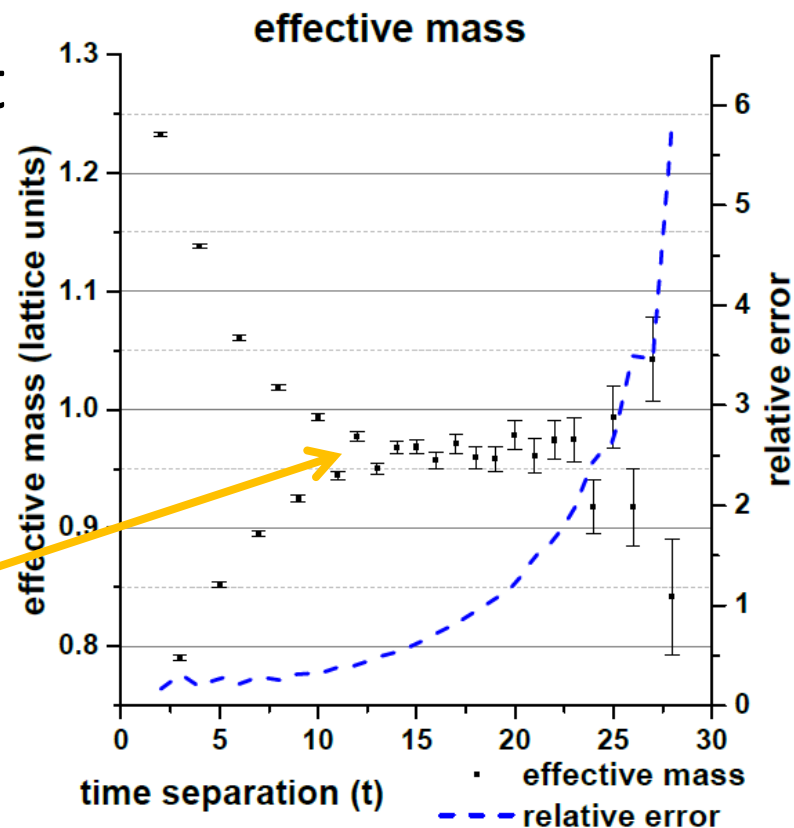
$$Z_0 \approx C(t)e^{E_0 t}$$

$$\langle 0 | \mathcal{O}_i | 0 \rangle \approx C_i(t_1, t_2) e^{-E_0(t_1+t_2)}$$

For large  $t$

It's 0.96!\*

\*Fermilab action does not tune rest mass, unlike RHQ



Taking ratios, and scaling data is sufficient to get a (crude) value for the matrix elements.

# Fit correlation functions (ii)

- 1) Robust error estimate through fitting
- 2) Fit towards higher signal region

Constrained curve fitting with Bayesian priors

parameters  $\rightarrow$  distributions

$$p_i \rightarrow \mu_i \pm \sigma_i$$

$$\chi^2 \rightarrow \chi^2 + \sum_i \frac{(\rho_i - \mu_i)^2}{\sigma_i^2}$$

Prior information guide fits  
Treated like data\*

Motivate priors!!!

# Motivate priors (ground states)

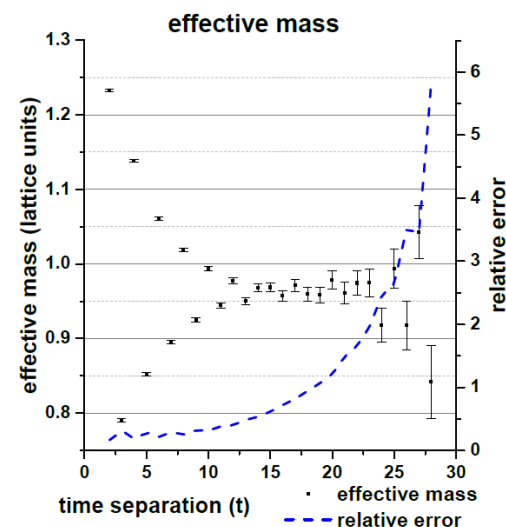
$$C^{2\text{pt}}(t) = \sum_n (-1)^{n(t+1)} \frac{Z_n^\dagger Z_n}{2E_n} \left( e^{-E_n t} + e^{-E_n(T-t)} \right)$$

$$C_i^{3\text{pt}}(t_1, t_2) = \sum_{m,n} (-1)^{n(t_2+1)} (-1)^{m(|t_1|+1)} \frac{\langle n | \mathcal{O}_i | m \rangle Z_n^\dagger Z_m}{4E_n E_m} e^{-E_n t_2} e^{-E_m |t_1|}$$

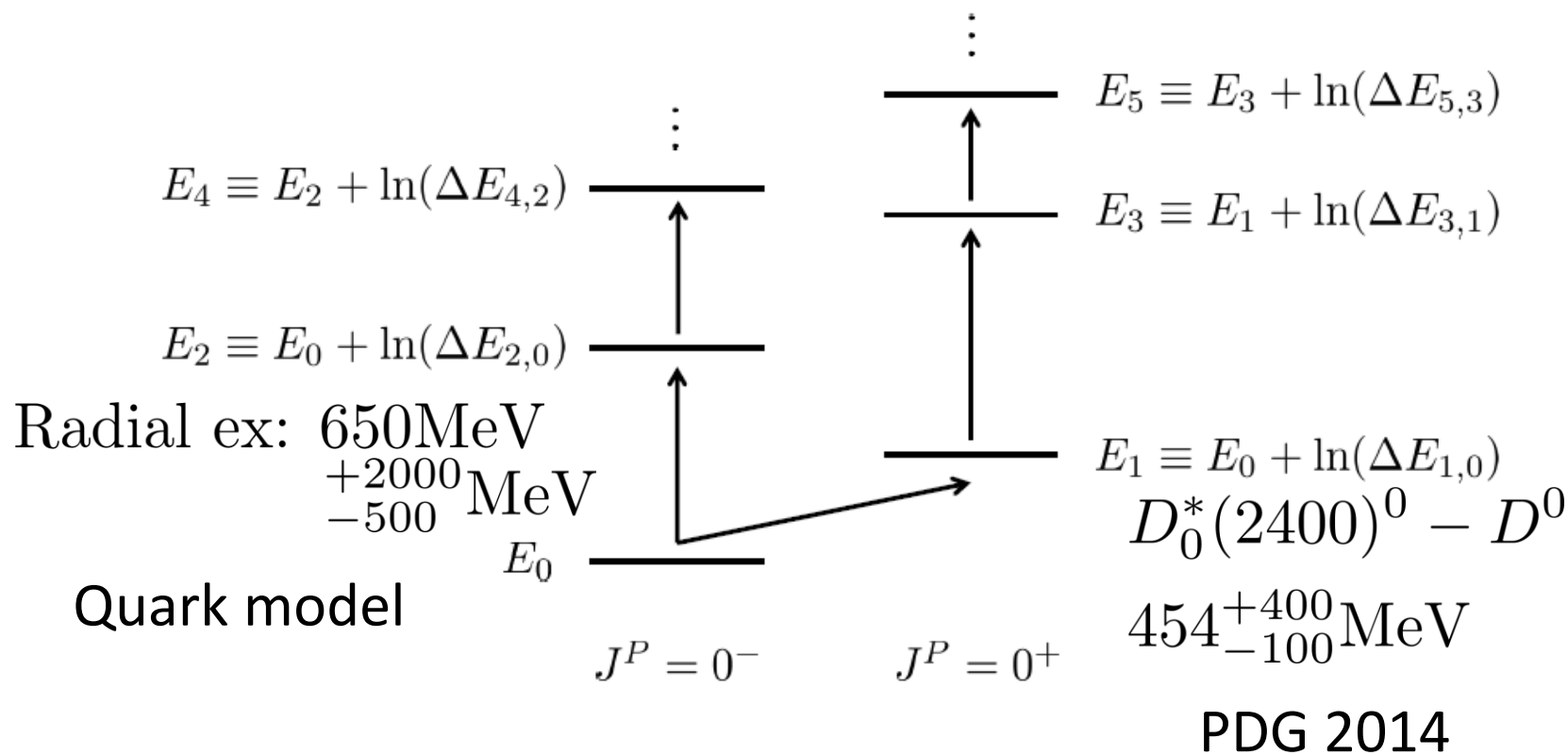
Want **data** to determine  $E_0, Z_0, \langle 0 | \mathcal{O}_i | 0 \rangle$

Ground state priors are unconstraining

Motivated by staring at the data



# Motivate priors (excited states)

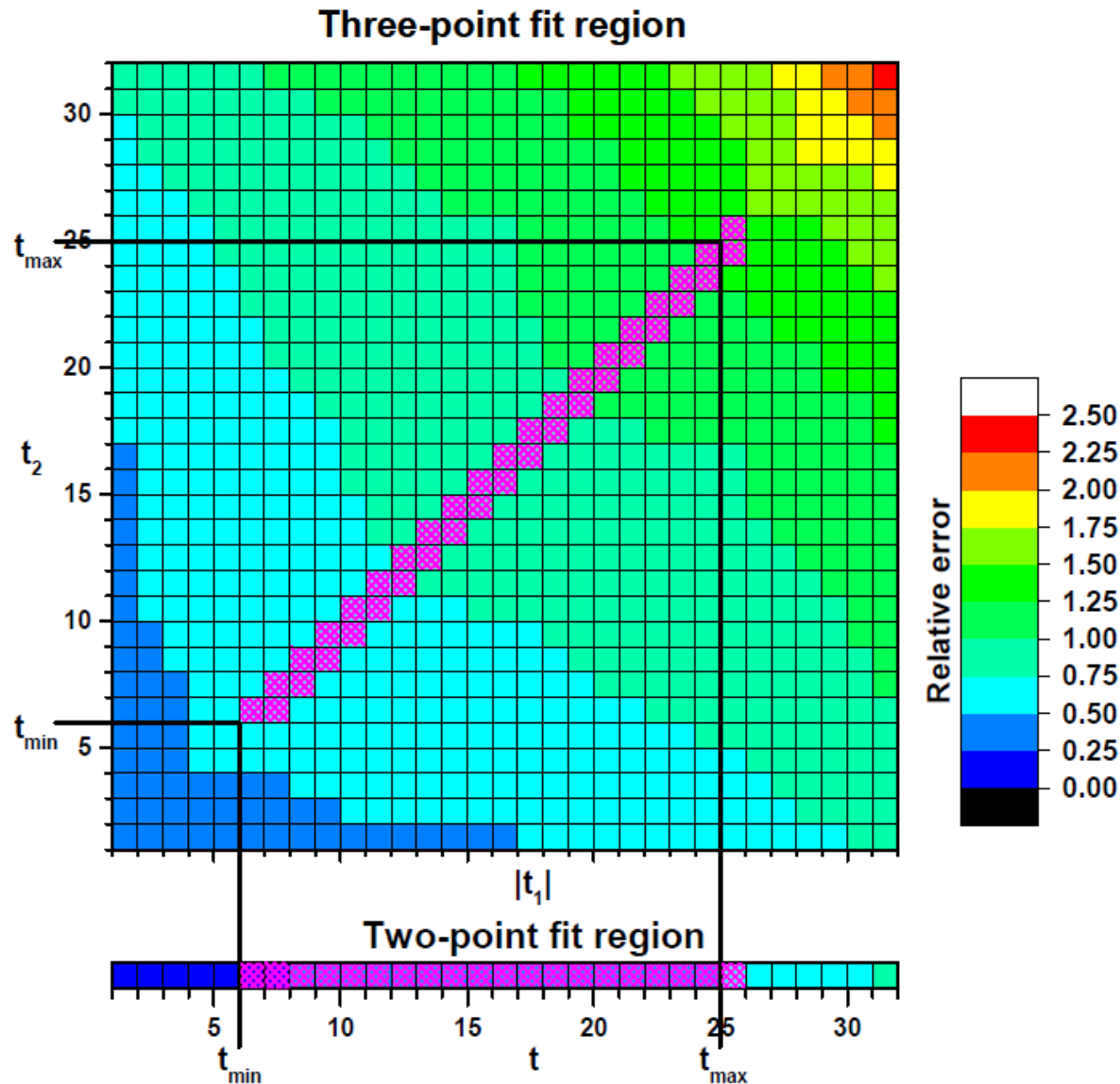


$$Z_n \simeq (0.5 \pm 1.0) Z_0$$

Heavy-quark smearing

$$\langle n | \mathcal{O}_i | m \rangle \simeq (0 \pm 1) \langle 0 | \mathcal{O}_i | 0 \rangle$$

# Correlator fit region

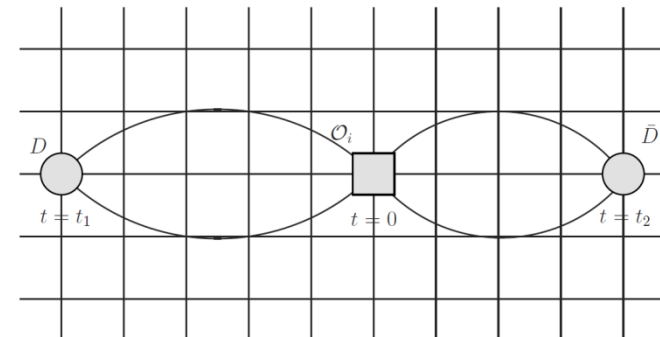


Simultaneous fit

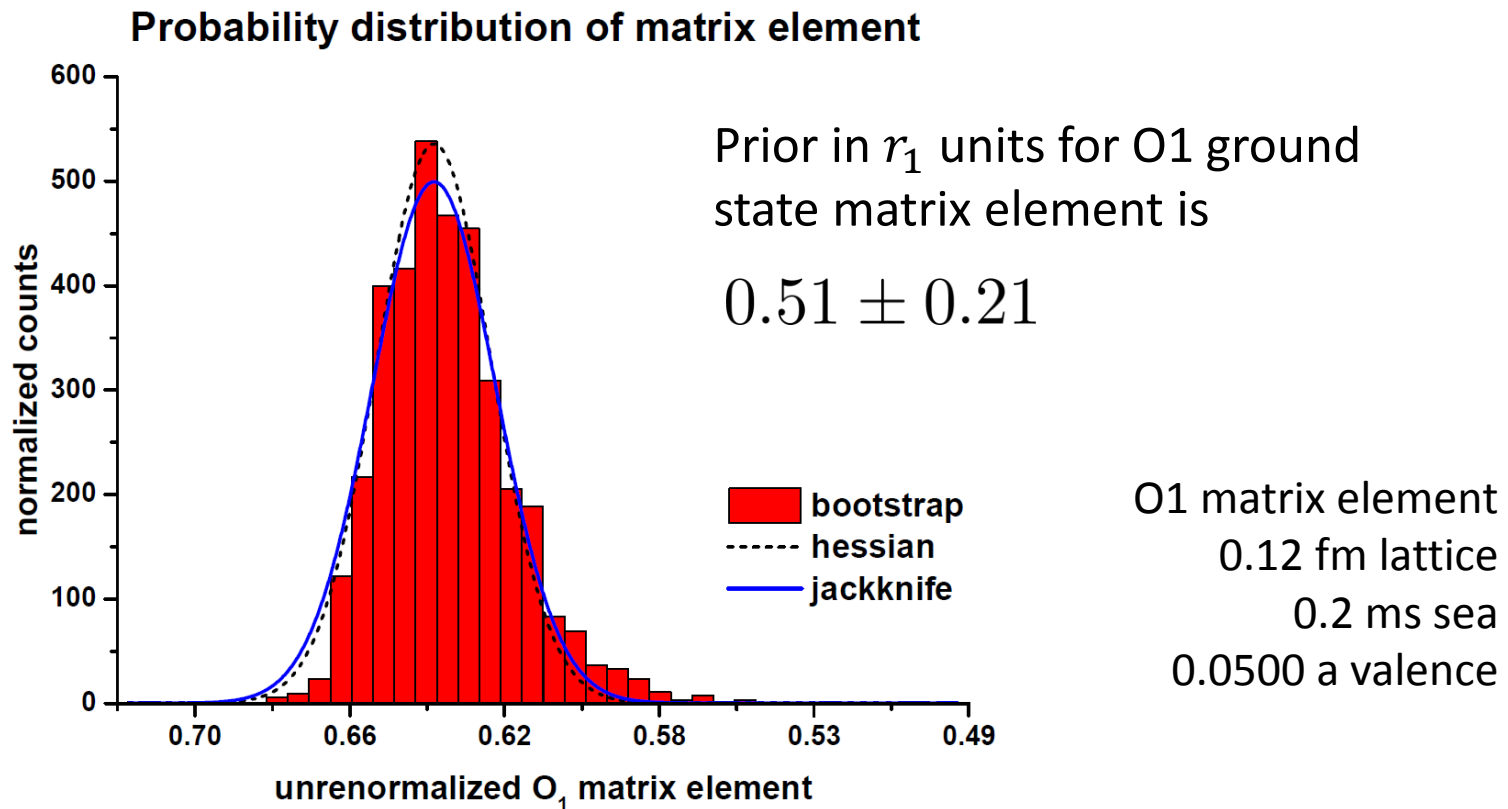
- Preserve correlation
- Disentangle  $Z_0$

Bi-diagonal fit

- Statistics limited
- Preserve  $0^+$  excited state information



# Fit quality



Distributions agree for the ground state matrix element fit parameter

- Priors are not constraining for this parameter

Stability plots for  $t_{min}$ ,  $t_{max}$ , smearing, # excited states in back up slides

# Mostly non-perturbative renorm.

- One-loop matching between lattice and continuum.
- Lattice regularization to  $\overline{MS}$ -NDR scheme at 3GeV
- BBGLN basis of Dirac operators

One-loop renormalization expression:

$$\langle \mathcal{O}_i \rangle^R = \boxed{Z_V^{hh} Z_V^{ll}} \left[ (1 + \alpha_s \boxed{\zeta_{ii}}) \langle \mathcal{O}_i \rangle^{\text{lat.}} + \boxed{\alpha_s \zeta_{ij} \langle \mathcal{O}_j \rangle^{\text{lat.}}} \right]$$

NP coefficients

Account for WF renorm.  
to all order

PT coefficient

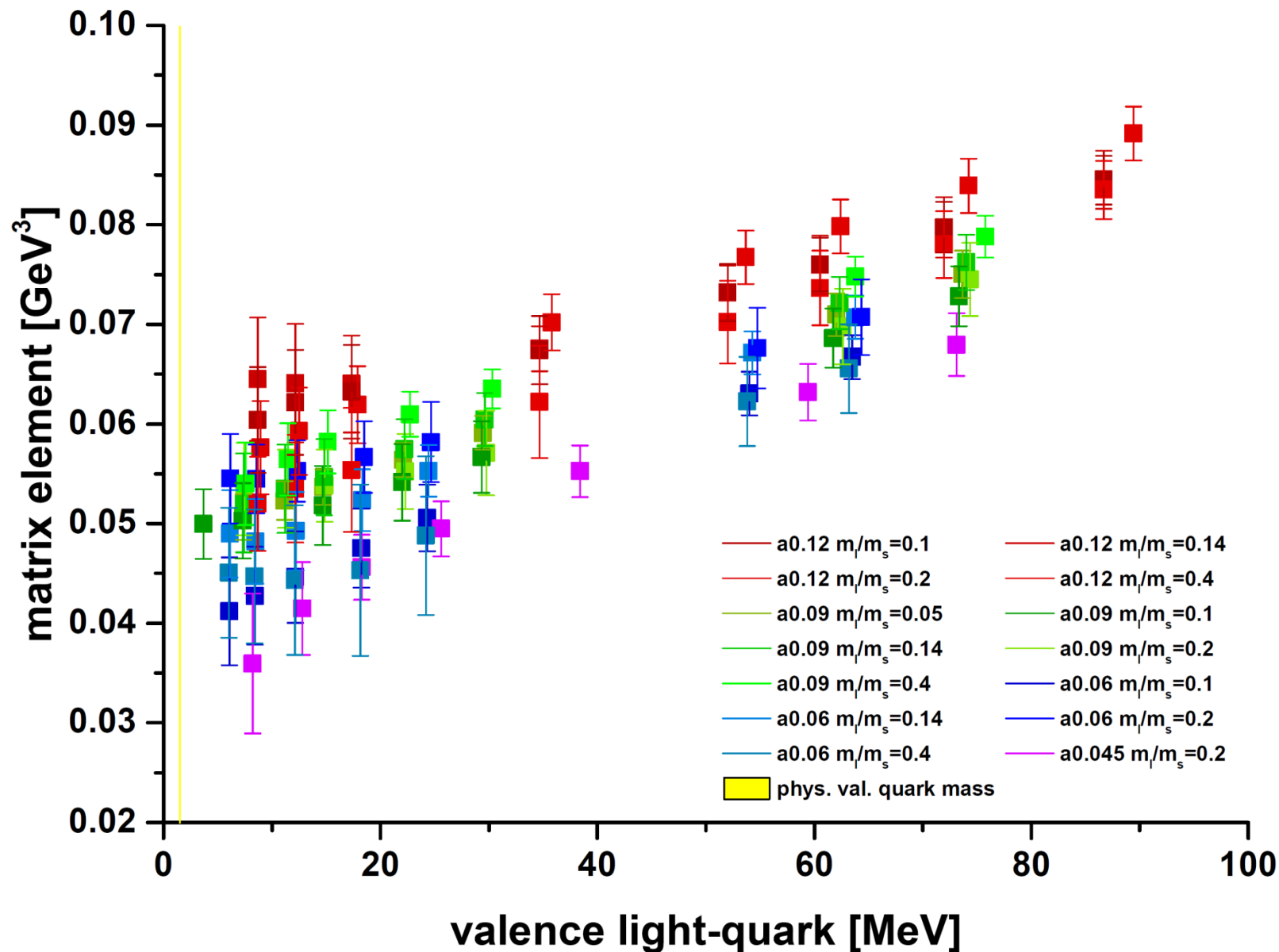
Account for vertex  
renormalization

Mixing under

renormalization

Errors start at  $O(\alpha_s^2, \alpha_s \Lambda_{\text{QCD}}/m_c)$

# Result of correlator fits



# FNAL/MILC

## *D*-meson mixing analysis

Chiral-continuum extrapolation  
and systematic error analysis

Fit function  
Stability of fit  
Error breakdown

# Chiral-continuum extrapolation

Extrapolate to physical point

Control systematic uncertainty

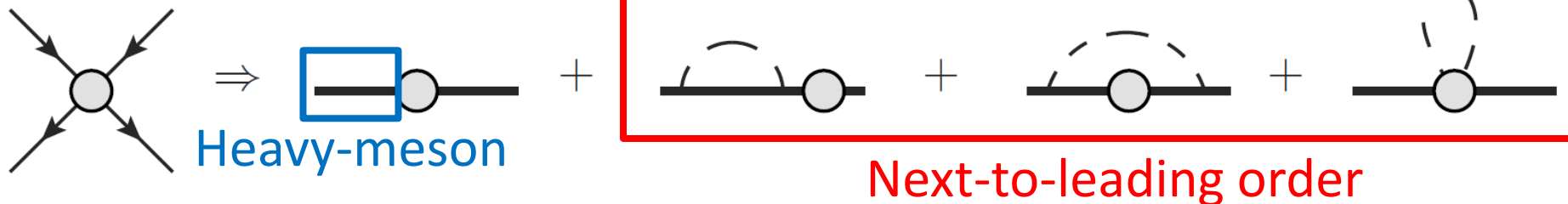
Effective theory lends understand to truncation errors

# Extrapolation to physical point

## Partially-quenched SU(3)

$$F_i^{\chi \text{ NLO}} = \beta_i \left( 1 + \frac{\mathcal{W}_{u\bar{c}} + \mathcal{W}_{c\bar{u}}}{2} + \mathcal{T}_u^{(i)} + \boxed{\tilde{\mathcal{T}}_u^{(i)}} + \text{analytic terms} \right) \\ + \beta'_i \left( \mathcal{Q}_u^{(i)} + \boxed{\tilde{\mathcal{Q}}_u^{(i)}} \right)$$

Staggered



$\tilde{\mathcal{T}}$  and  $\tilde{\mathcal{Q}}$  are the NLO wrong-spin taste-mixing terms

Wrong-spin because the Dirac structure is in general different from  $\mathcal{O}_i$

Taste-mixing because taste-index between the two bilinears are summed over

Copy-mixing also, but copy symmetry is exact

# Systematic error analysis

Sources of systematic error:

Chiral logarithms  
Truncation errors

} Chiral-continuum extrapolation

Heavy-quark discretization

Renormalization

Heavy-quark and light-quark masses

Finite volume

Scale error

} Other systematics

Bayesian statistics treat systematics like statistical error

# Analytic terms and HM $\chi$ PT

**Chiral fit function includes NNLO analytic terms.**

At NLO:  $a^2$  term  
At NNLO:  $a^4$  term

} light quark and gluon discretization error

NNLO mass dependent terms accounts for NNLO chiral logarithms truncation.

$$F_{\text{analytic}} = \sum_j c_j P_j(m_u, m_l, m_s, a^2)$$

**Work at leading order in HM $\chi$ PT**

Option to include leading  $1/M_D$  errors in  $\chi$ PT

# Heavy quark discretization errors

## Operator improvement

$$\Psi(x) = \boxed{e^{M_1 a/2} \left[ 1 + a d_1 \boldsymbol{\gamma} \cdot \boldsymbol{D} \right]} + \frac{1}{2} a^2 \left( d_2 \Delta^{(3)} + i d_B \boldsymbol{\Sigma} \cdot \boldsymbol{B} + d_E \boldsymbol{\alpha} \cdot \boldsymbol{E} \right) \psi(x)$$

“Heavy-quark rotation”  
Operator & action both  
tree-level  $a$  improved

Tree-level  $a^2$

[9604004]

$d$ s are adjusted by matching lattice and continuum spinors.

Action @  $a^2, a^3$  from Oktay Kronfeld (2008) [0803.0523]

$\alpha_s a$  corrections are estimated

Main result of the Massive Fermions paper:

Matching finite @  $am_0 \rightarrow 0$  and zero @  $am_0 \rightarrow \infty$

$$F_{\text{HQ disc.}} = \sum_i z_i (a \Lambda_{\text{HQ}})^{s_i} f_i(m_0 a)$$

# Renormalization errors

Fit for  $\alpha_s^2$  renormalization errors. HQ errors include  $\alpha_s \Lambda_{\text{QCD}}/m_c$ .

$$\begin{aligned} \langle D | \mathcal{O}_i | \bar{D} \rangle^R = & Z_V^{hh} Z_V^{ll} \left[ (1 + \alpha_s \zeta_{ii} + \alpha_s^2 \xi_{ii} + \mathcal{O}(\alpha_s^3)) \langle D | \mathcal{O}_i | \bar{D} \rangle \right. \\ & \left. + (\alpha_s \zeta_{ij} + \alpha_s^2 \xi_{ij} + \mathcal{O}(\alpha_s^3)) \langle D | \mathcal{O}_j | \bar{D} \rangle \right] \end{aligned}$$

Renormalize data:  $\langle \mathcal{O}_i \rangle^R = Z_V^{hh} Z_V^{ll} \left[ (1 + \alpha_s \zeta_{ii}) \langle \mathcal{O}_i \rangle^{\text{lat.}} + \alpha_s \zeta_{ij} \langle \mathcal{O}_j \rangle^{\text{lat.}} \right]$

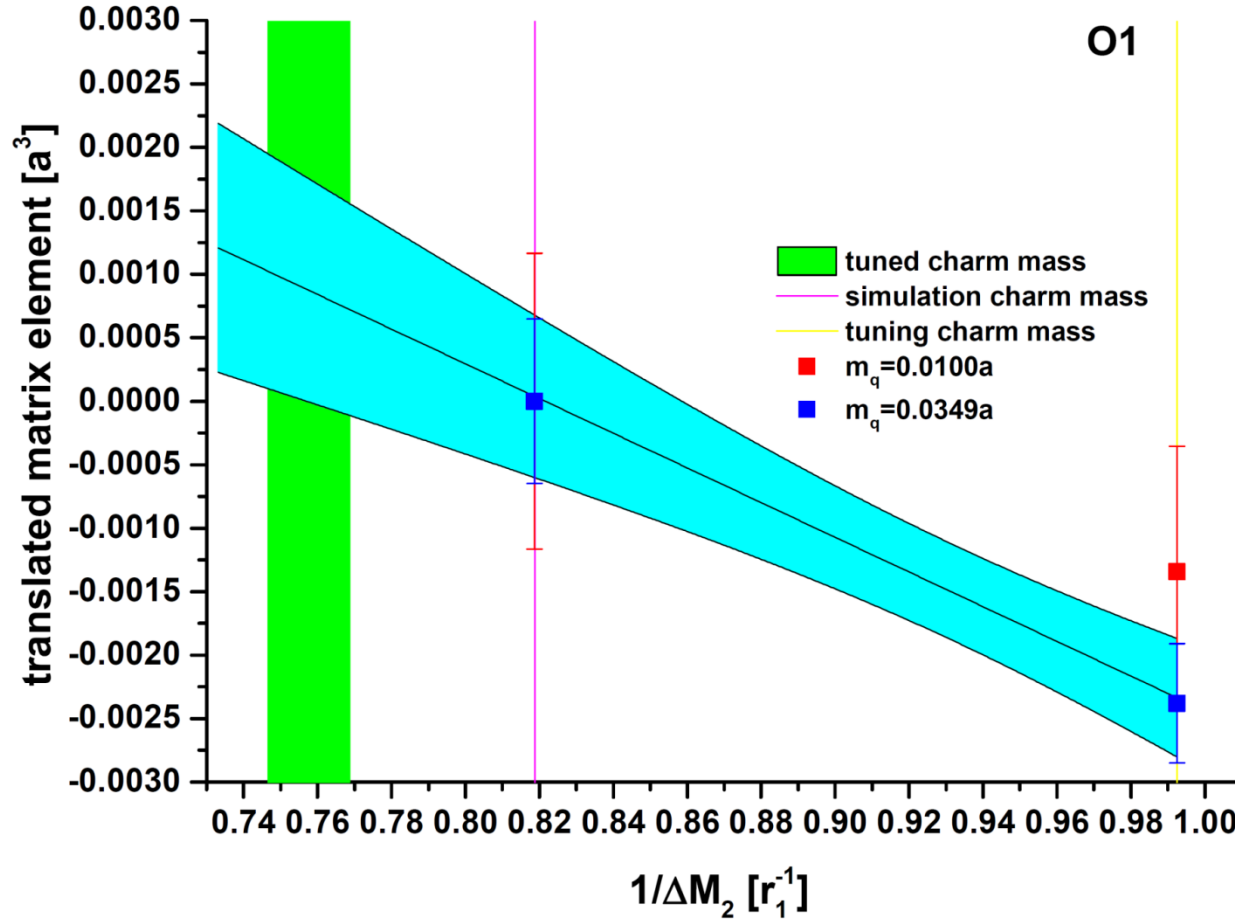
Fix scale, scheme, evanescent operators  $\zeta \simeq \mathcal{O}(1)$

Fit  $\alpha_s^2$ :  $F_i^{\text{renorm}} = Z_V^{hh} Z_V^{ll} \left[ \alpha_s^2 \xi_{ii} \langle D | \mathcal{O}_i | \bar{D} \rangle + \alpha_s^2 \xi_{ij} \langle D | \mathcal{O}_j | \bar{D} \rangle \right]$

For same scale, scheme, etc... expect  $\xi = 0 \pm 1$

Power counting  $\sim 6.4\%$

# Heavy-quark tuning



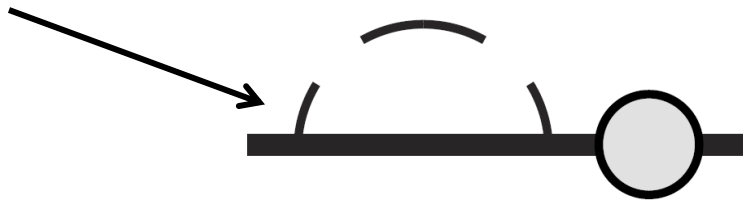
Tune  $D_S$  for  $m_c$

Perform the shift  $F_i^\kappa = -\sigma_i \times 1/\Delta M_2$  @ the level of ChiPT  
 $\sigma_i$  and  $1/\Delta M_2$  are introduced as priors

# Parametric errors

A list of the largest parametric errors

$g_{DD^*\pi}$



From  $D^* \rightarrow D\pi$  studies  
 $DD^*$  form doublet under HQ spin symmetry

$r_1/a$

Errors with correlations are included

$r_1$

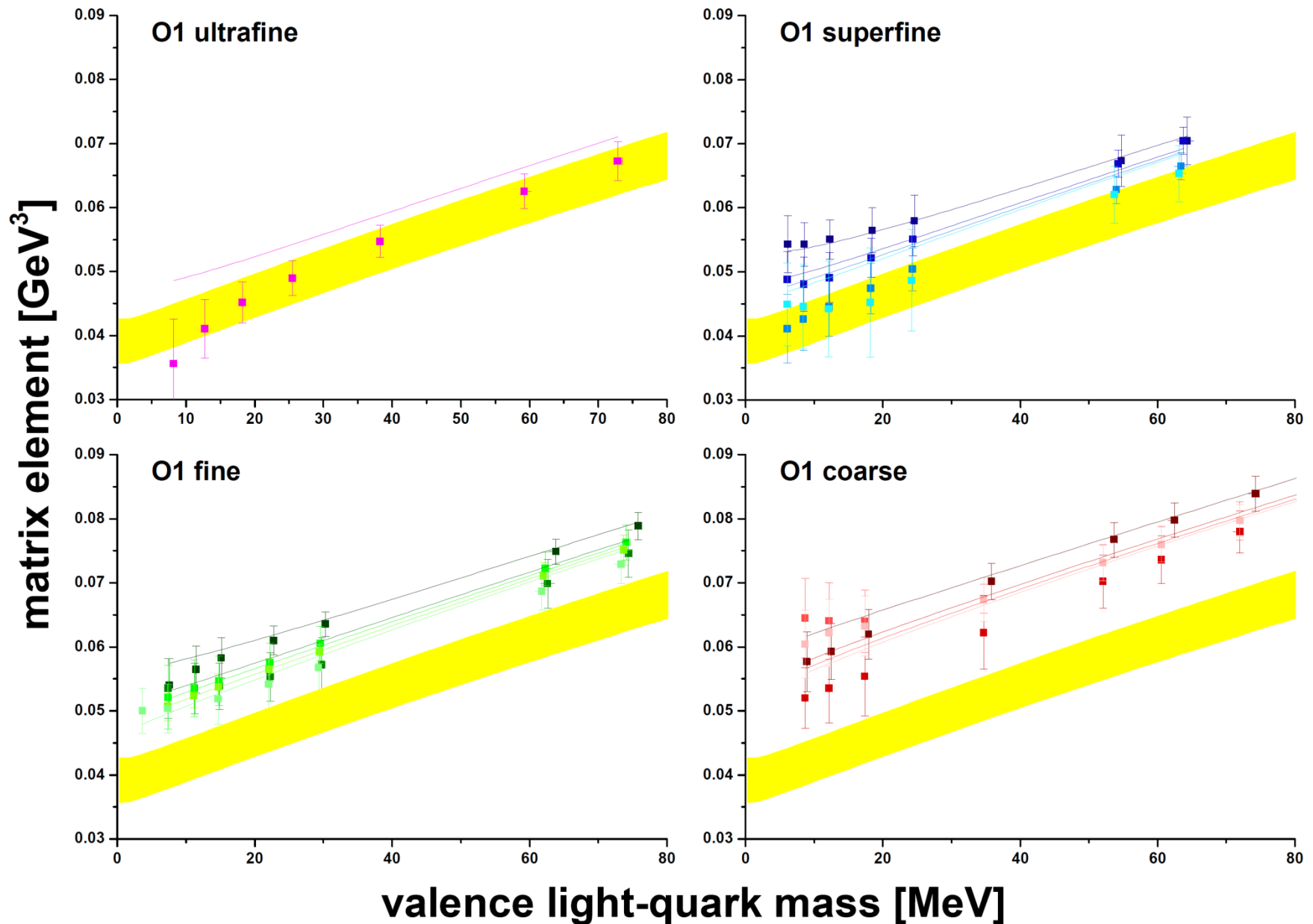
# Final fit function

$$\begin{aligned} F_i = & F_i^{\text{NLO}} \chi^{\text{PT}} + F_i^{\text{NNLO}} \text{analy.} \\ & + F_i^{\text{HQ}} + F_i^{\alpha_s^2} \text{renorm.} - F_i^{\kappa} \\ & + \text{finite volume correction} \\ & + \text{parametric errors} \end{aligned}$$

(will) Fit to all 5 operators to preserve correlations  
Final error budget is a covariance matrix

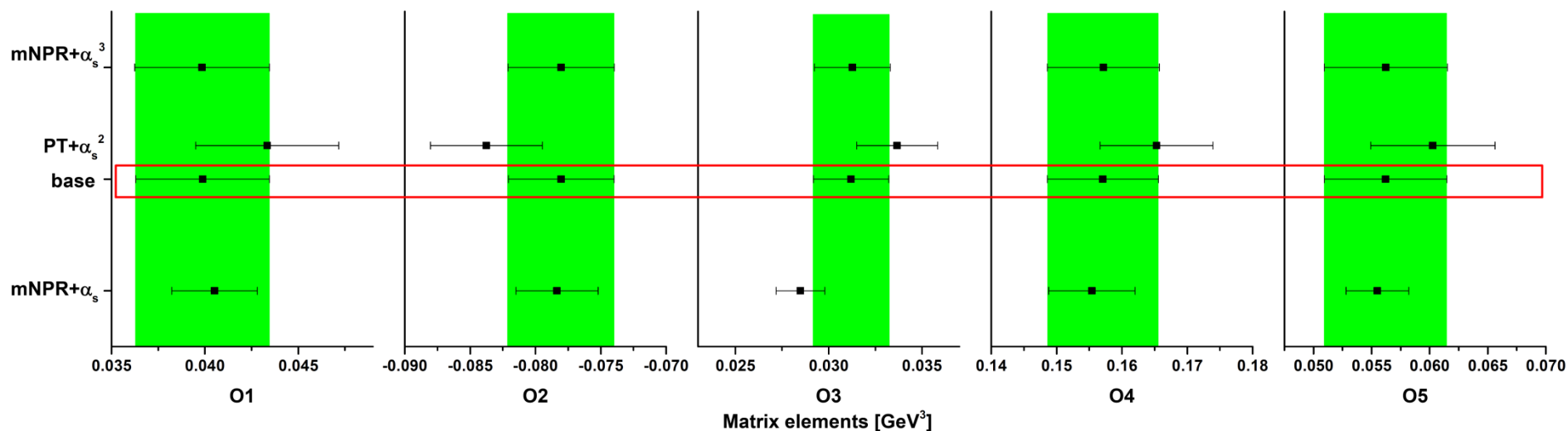
This fit accounts for every source of error  
that we would like to include

# Chiral-continuum extrapolation



# Renormalization stability plot

$$\langle D|\mathcal{O}_i|\bar{D}\rangle^R = Z_V^{hh} Z_V^{ll} \left[ (1 + \alpha_s \zeta_{ii} + \alpha_s^2 \xi_{ii} + \mathcal{O}(\alpha_s^3)) \langle D|\mathcal{O}_i|\bar{D}\rangle + (\alpha_s \zeta_{ij} + \alpha_s^2 \xi_{ij} + \mathcal{O}(\alpha_s^3)) \langle D|\mathcal{O}_j|\bar{D}\rangle \right]$$



Error bar increases with  $\alpha_s^2$  terms in fit

Fit remains unchanged when adding  $\alpha_s^3$  terms in fit

One-loop contribution  $\sim 20\%$

Power counting error estimate  $\sim 6.5\%$

Error from fit 3 to 6%. Operator dependent.

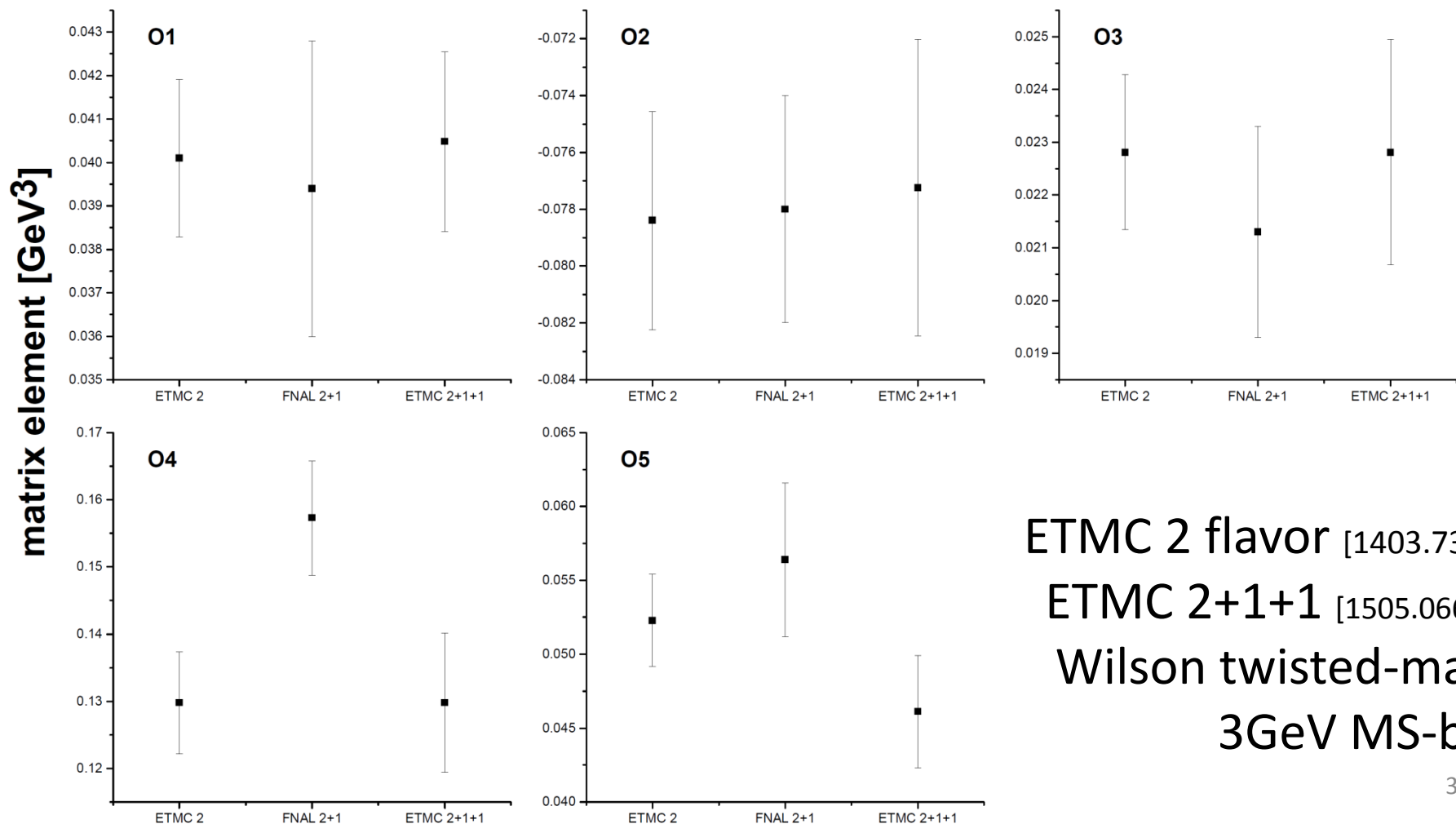
# Preliminary error budget

	$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_4$	$\mathcal{O}_5$
Statistical	4.2%	2.4%	3.8%	2.7%	4.7%
Total $\chi$ -cont. err.	2.3%	2.4%	2.3%	1.3%	3.3%
Heavy-quark disc.	1.9%	1.8%	1.4%	2.1%	1.6%
Renormalization	5.7%	3.0%	3.9%	3.2%	6.5%
HQ mistuning	0.1%	0.0%	0.0%	0.0%	0.0%
LQ mass uncert.	0.3%	0.6%	0.2%	0.5%	0.3%
$r_1/a$	1.2%	1.4%	1.6%	1.6%	2.0%
$r_1$	2.1%	2.1%	2.1%	2.1%	2.1%
Total error	8.1%	5.2%	6.7%	5.6%	9.3%

Goal of 10% total error to match projected  
experimental error for the next decade

# Preliminary results

	$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_4$	$\mathcal{O}_5$
$\frac{\langle D \mathcal{O}_i \bar{D}\rangle}{m_D}$ [GeV <sup>3</sup> ]	0.039(3)	-0.079(4)	0.031(2)	0.155(9)	0.055(5)



ETMC 2 flavor [1403.7302]  
 ETMC 2+1+1 [1505.06639]  
 Wilson twisted-mass  
 3GeV MS-bar

# Outlook

Paper!

Bag parameters w/ Ethan

No plans for HISQ. Errors are good.

A photograph of the University of Minnesota's Campanile tower, a large, white, hourglass-shaped building with a central glass section, standing prominently on the left. To the right, a tall, thin, white sculpture, known as the 'Fountain of Time', rises from a frozen pond. The pond's surface is covered in ice with some open water reflecting the sky. The background features a sunset sky with orange and blue hues, and silhouettes of trees and a flagpole. A blue banner with the text 'Thank you' is overlaid on the bottom left.

Thank you

# Data

# MILC asqtad ensembles

$a(fm)$	$\left(\frac{L}{a}\right)^3 \times \frac{T}{a}$	$m_\pi L$	$am_l/am_s$	$m_\pi(\text{MeV})$	$N_{\text{confs}}$	$r_1/a$
0.12	$24^3 \times 64$	3.84	0.1	274	2099	2.647
0.12	$20^4 \times 64$	3.78	0.14	325	2110	2.635
0.12	$20^4 \times 64$	6.27	0.2	388	2259	2.618
0.12	$20^4 \times 64$	6.22	0.4	557	2052	2.644
0.09	$64^3 \times 64$	4.80	0.05	176	791	3.691
0.09	$40^4 \times 64$	4.21	0.1	249	1015	3.695
0.09	$32^4 \times 64$	4.11	0.14	308	984	3.697
0.09	$28^4 \times 64$	4.14	0.2	354	1931	3.699
0.09	$28^4 \times 64$	5.78	0.4	506	1996	3.712
0.06	$64^3 \times 144$	4.27	0.1	223	827	5.281
0.06	$56^4 \times 144$	4.39	0.14	264	801	5.292
0.06	$48^4 \times 144$	4.49	0.2	317	673	5.296
0.06	$48^4 \times 144$	6.33	0.4	451	593	5.283
0.045	$64^3 \times 192$	4.56	0.2	323	801	7.115

# Valence light-quark parameters

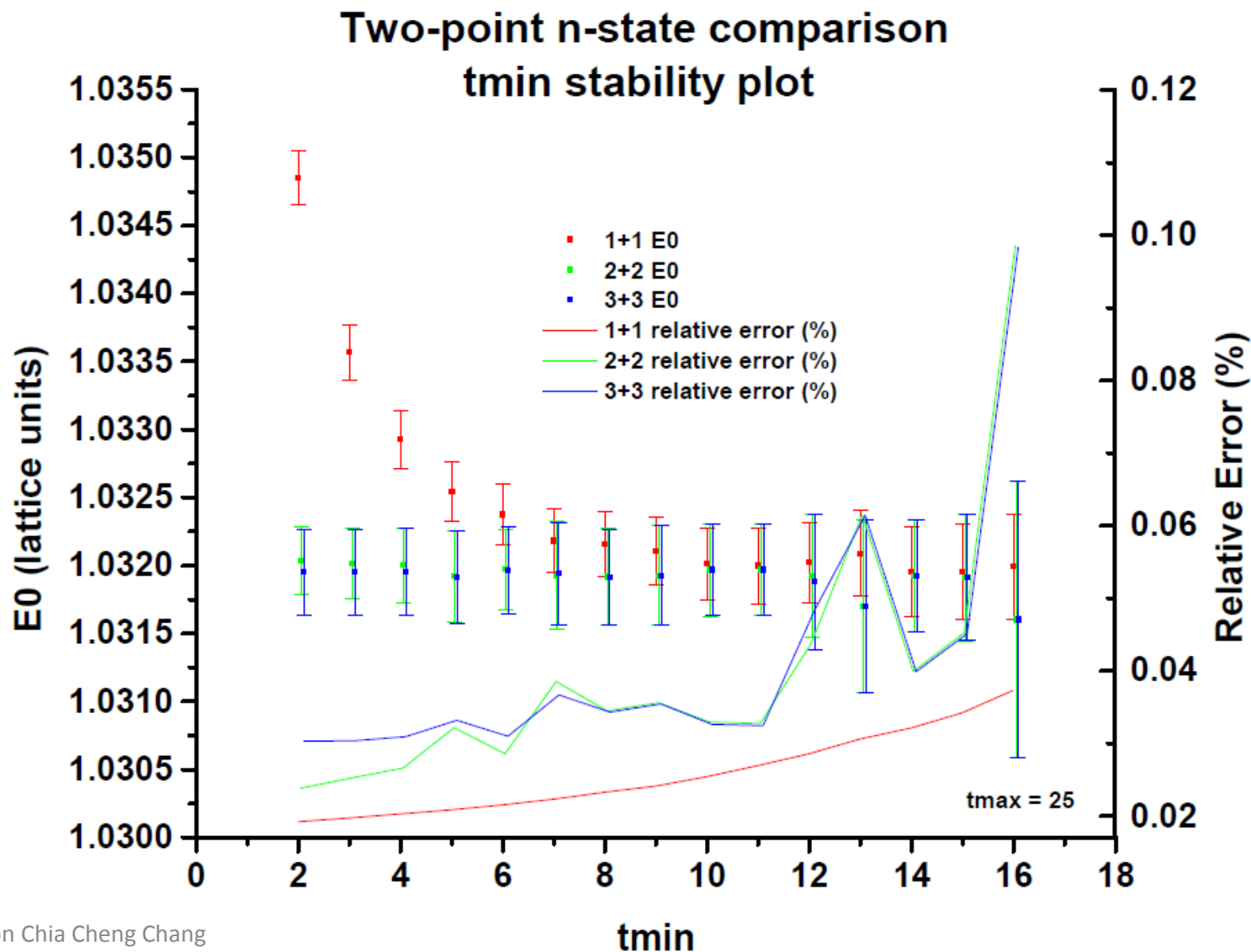
$a(fm)$	$am_l/am_s$	$am_q(\text{lattice units})$
0.12	0.1—0.4	0.0050, 0.0070, 0.0100, 0.0200, 0.0300, 0.03497, 0.0415, 0.0500
0.09	0.05	0.00155, 0.0031, 0.0062, 0.0093, 0.0124, 0.0261, 0.0310
0.09	0.1—0.4	0.0031, 0.0047, 0.0062, 0.0093, 0.0124, 0.0261, 0.0310
0.06	0.1—0.4	0.0018, 0.0025, 0.0036, 0.0054, 0.0072, 0.0160, 0.0188
0.045	0.2	0.0018, 0.0028, 0.0040, 0.0056, 0.0084, 0.0130, 0.0160

# Valence heavy-quark parameters

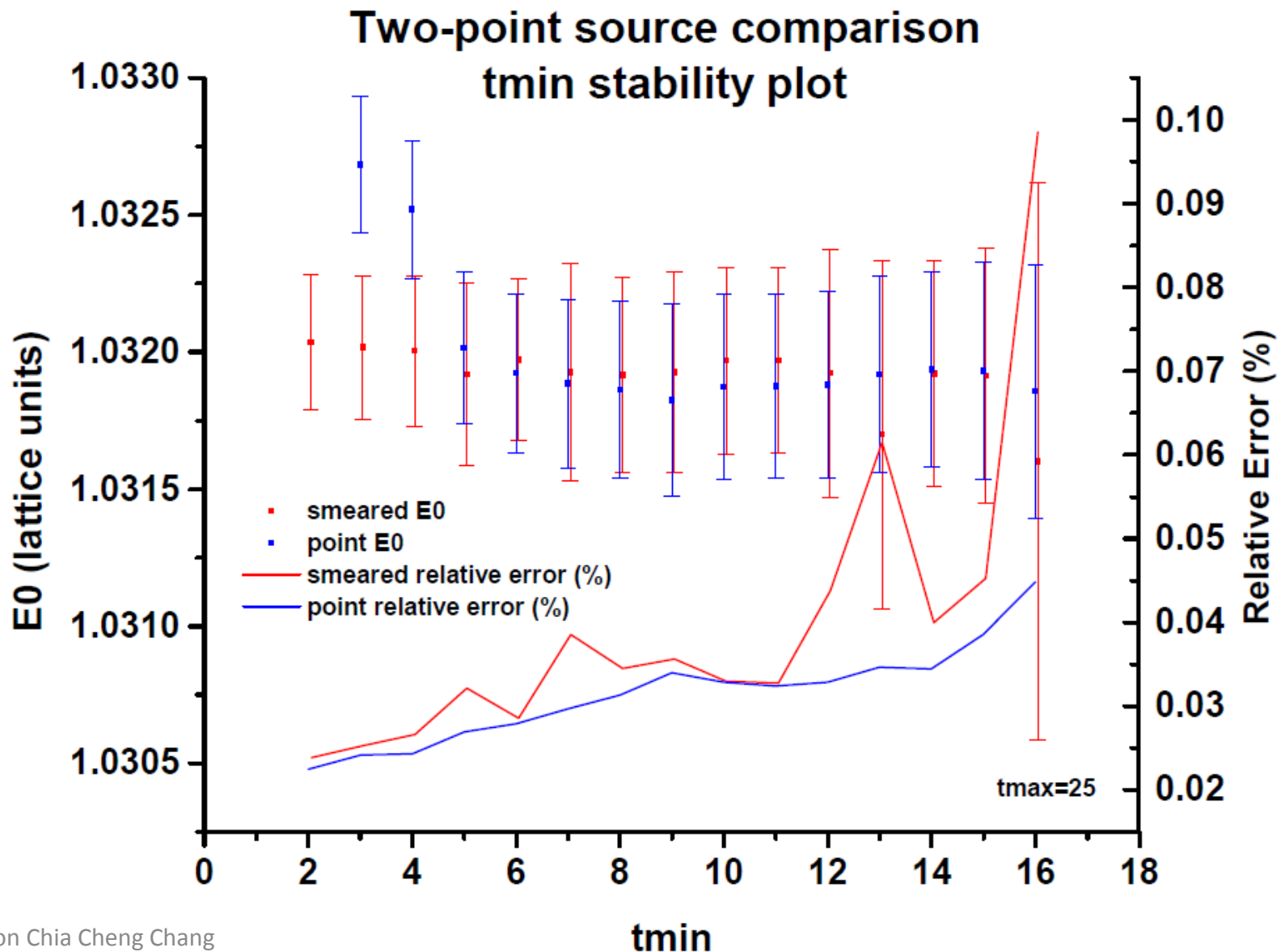
$a(\text{fm})$	$m_l/m_s$	$\kappa_{\text{crit}}$	$\kappa_{\text{tune}}$	$\kappa_{\text{sim.}}$	$u_0$	$r_1/a$
0.12	0.4	0.14073	0.12452(15)(16)	0.1259	0.8688	2.821123
0.12	0.2	0.14091	0.12423(15)(16)	0.1254	0.8677	2.738591
0.12	0.14	0.14095	0.12423(15)(16)	0.1254	0.8678	2.738591
0.12	0.1	0.14096	0.12423(15)(16)	0.1254	0.8678	2.738591
0.09	0.4	0.139052	0.12737(9)(14)	0.1277	0.8788	3.857729
0.09	0.2	0.139119	0.12722(9)(14)	0.1276	0.8782	3.788732
0.09	0.14	0.139134	0.12718(9)(14)	0.1275	0.8781	3.771633
0.09	0.1	0.139173	0.12714(9)(14)	0.1275	0.8779	3.754593
0.09	0.05	0.13919	0.12710(9)(14)	0.1275	0.877805	3.737613
0.06	0.4	0.137582	0.12964(4)(11)	0.1295	0.8881	5.399129
0.06	0.2	0.137632	0.12960(4)(11)	0.1296	0.88788	5.353063
0.06	0.14	0.137667	0.12957(4)(11)	0.1296	0.88776	5.330159
0.06	0.1	0.137678	0.12955(4)(11)	0.1296	0.88764	5.307340
0.045	0.2	0.13664	0.130921(16)(70)	0.1310	0.89511	7.208234

# Correlator stability

# Correlator fit: n-states

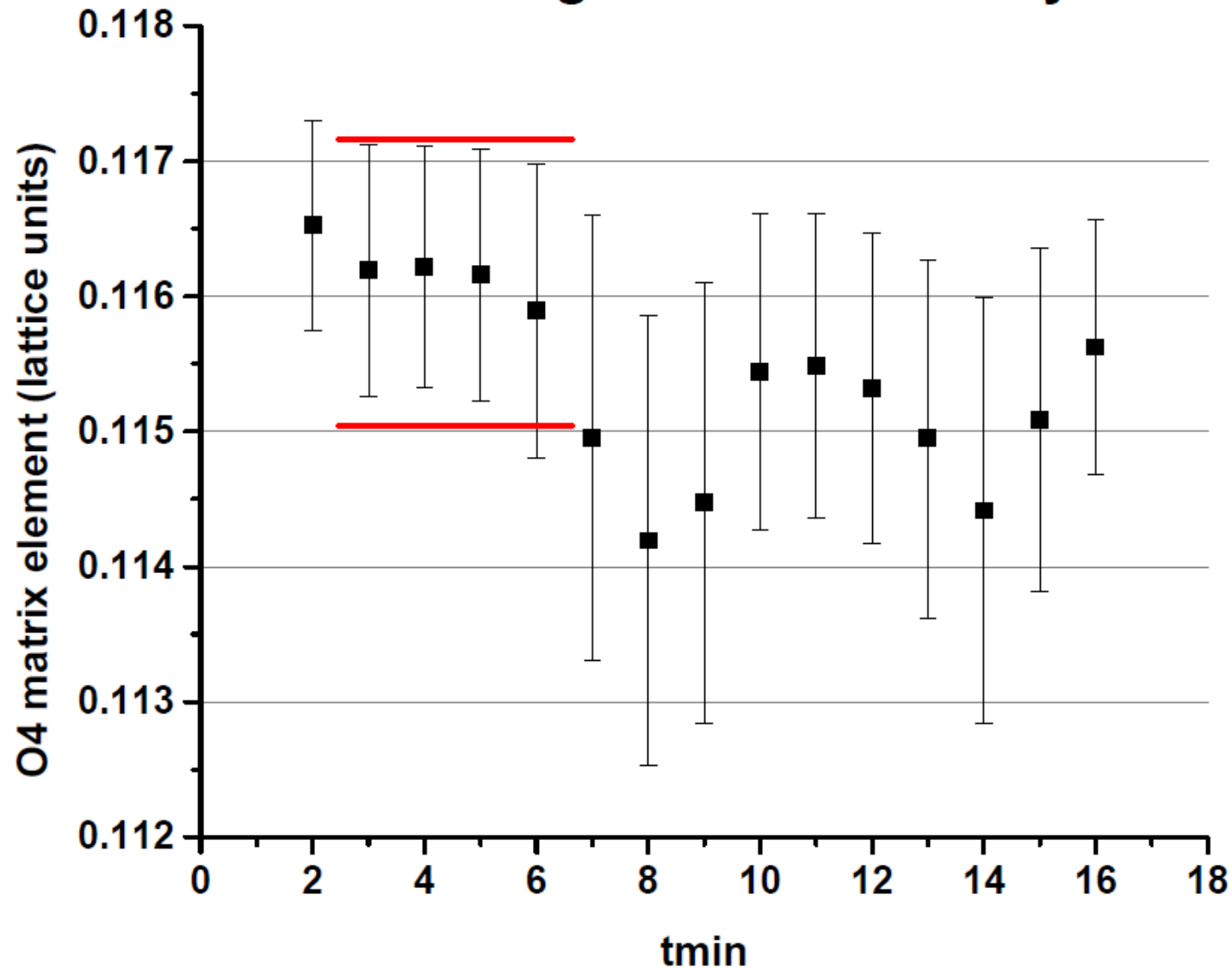


# Correlator fit: smearing

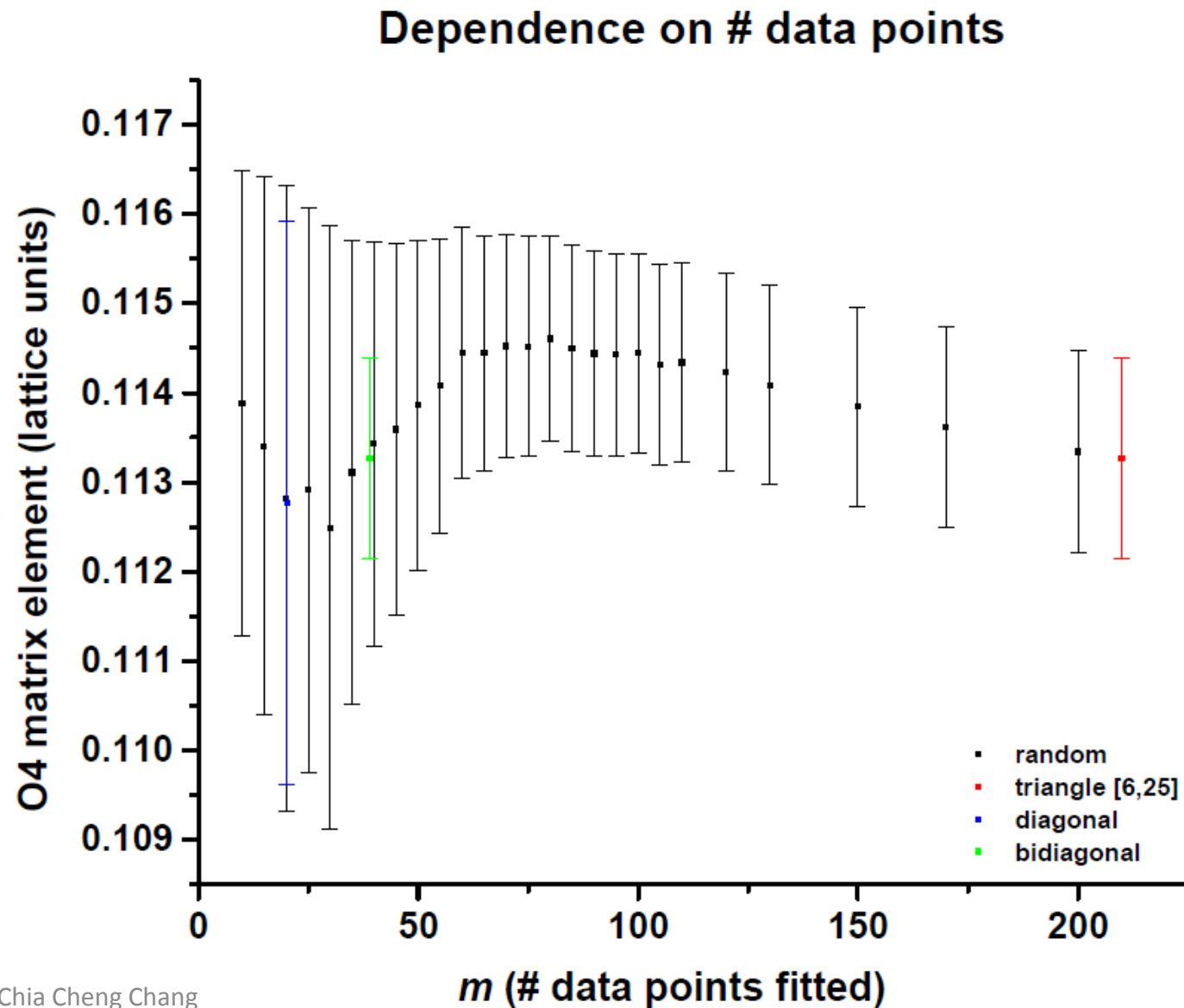


# Correlator fit: matrix element

## O4 bidiagonal tmin stability



# Correlator fit: random sampling



# ChiPT stability

# Analytic terms and HM $\chi PT$

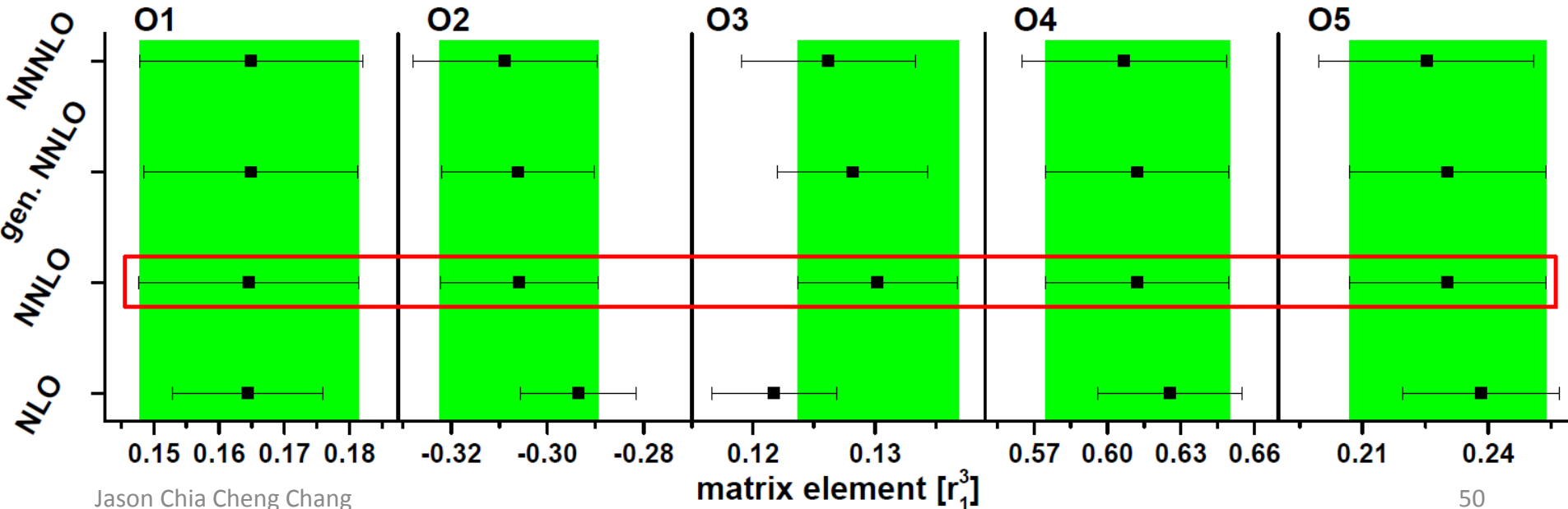
$$F_i^{\text{pref.}} = F_i^{\text{logs}} + F^{\text{NLO}} + F^{\text{NNLO}} + F_i^{\text{HQ error}} - F_i^{\kappa\text{-tune}} - F_i^{\text{renorm}}$$

$$F_i^{\text{NLO}} = F_i^{\text{pref.}} - F^{\text{NNLO}}$$

$$F_i^{\text{NNLO}} = F_i^{\text{pref.}}$$

$$F_i^{\text{gen. NNLO}} = F_i^{\text{pref.}} + F\alpha_s a^2$$

$$F_i^{\text{NNNLO}} = F_i^{\text{pref.}} + F^{\text{NNNLO}}$$



# Heavy quark discretization errors

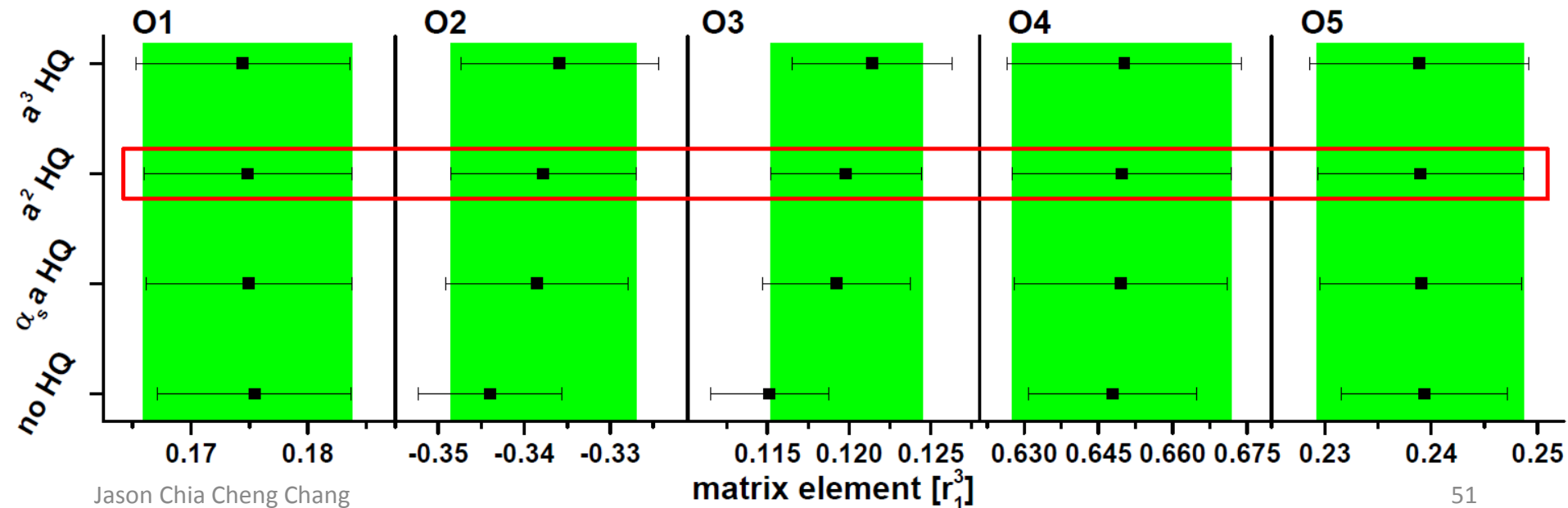
$$F_i^{\text{pref.}} = F_i^{\text{logs}} + F^{\text{NLO}} + F^{\text{NNLO}} + F_i^{\text{HQ error}} - F_i^{\kappa\text{-tune}} - F_i^{\text{renorm}}$$

$$F_i^{\text{no HQ}} = F_i^{\text{pref.}} - F_i^{\text{HQ error}}(a) - F_i^{\text{HQ error}}(a^2)$$

$$F_i^{\alpha_s a \text{ HQ}} = F_i^{\text{pref.}} - F_i^{\text{HQ error}}(a^2)$$

$$F_i^{a^2 \text{ HQ}} = F_i^{\text{pref.}}$$

$$F_i^{a^3 \text{ HQ}} = F_i^{\text{pref.}} + F_i^{\text{HQ error}}(a^3)$$



# Renormalization errors

$$F_i^{\text{pref.}} = F_i^{\text{logs}} + F^{\text{NLO}} + F^{\text{NNLO}} + F_i^{\text{HQ error}} - F_i^{\kappa\text{-tune}} - F_i^{\text{renorm}}$$

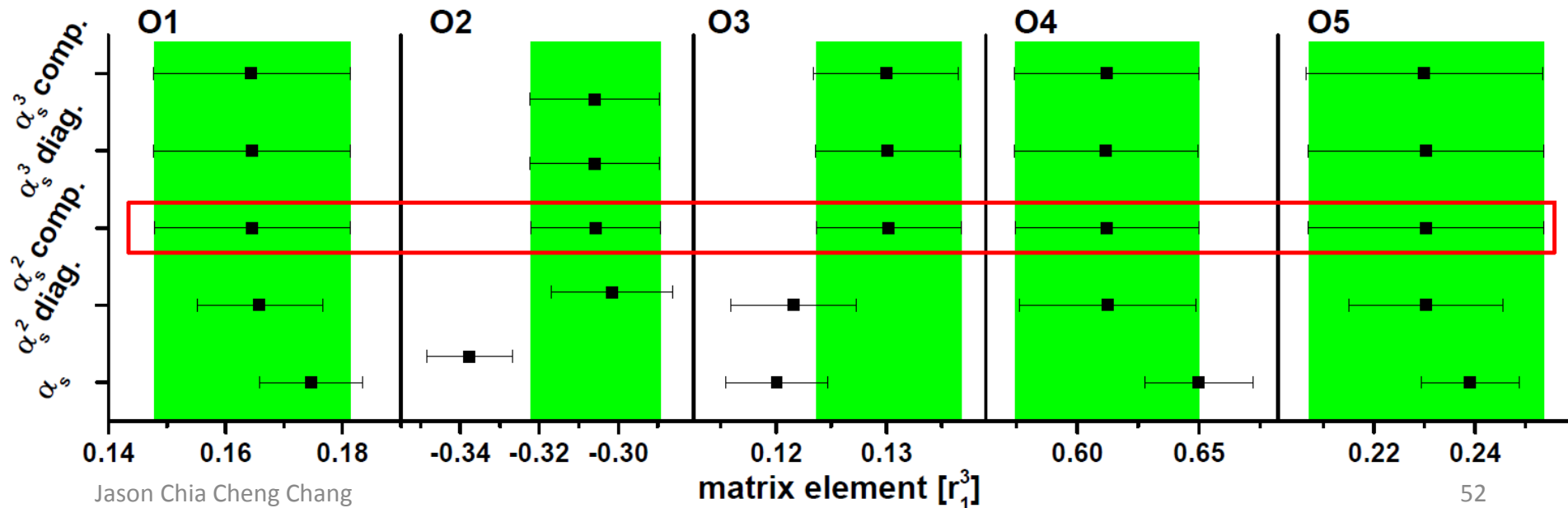
$$F_i^{\alpha_s} = F_i^{\text{pref.}} - F_i^{\xi_{ii}} - F_i^{\xi_{ij}}$$

$$F_i^{\alpha_s^2 \text{ diag.}} = F_i^{\text{pref.}} + F_i^{\xi_{ij}}$$

$$F_i^{\alpha_s^2 \text{ comp.}} = F_i^{\text{pref.}}$$

$$F_i^{\alpha_s^3 \text{ diag.}} = F_i^{\text{pref.}} - F_i^{\psi_{ii}}$$

$$F_i^{\alpha_s^3 \text{ comp.}} = F_i^{\text{pref.}} - F_i^{\psi_{ii}} - F_i^{\psi_{ij}}$$



# Data cuts in chiral extrapolation

Preferred:  $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}$  and  $\{\mathcal{O}_4, \mathcal{O}_5\}$  simultaneous w/ all data

Individual: 5 operators fit individually

$m_{\text{val}} < 560\text{MeV}$ : Drop valence quarks around  $\rho$  mass

$a < 0.12\text{fm}$ : Drops 0.12fm ensembles (check continuum extrap.)

